ISRAEL JOURNAL OF MATHEMATICS **131** (2002), 333–359

FIBREWISE CONSTRUCTION APPLIED TO LUSTERNIK–SCHNIRELMANN CATEGORY

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ABSTRACT

In this paper a variant of Lusternik–Schnirelmann category is presented which is denoted by $Qcat(X)$. It is obtained by applying a base-point free version of $Q = \Omega^\infty\Sigma^\infty$ fibrewise to the Ganea fibrations. We prove $cat(X) \geq Qcat(X) \geq \sigma cat(X)$ where $\sigma cat(X)$ denotes Y. Rudyak's strict category weight. However, $Qcat(X)$ approximates $cat(X)$ better, because, e.g., in the case of a rational space $Qcat(X) = cat(X)$ and $\sigma cat(X)$ equals the Toomer invariant.

We show that $Qcat(X \times Y) \leq Qcat(X) + Qcat(Y)$. The invariant $Qcat$ is designed to measure the failure of the formula $cat(X \times S^r) = cat(X) + 1$. In fact for 2-cell complexes $Qcat(X) < cat(X) \Leftrightarrow cat(X \times S^r) = cat(X)$ for some $r \geq 1$.

Received November 2, 2000 and in revised form August 17, 2001

We note that the paper is written in the more general context of a functor λ from the category of spaces to itself satisfying certain conditions; $\lambda = Q$, $\Omega^n\Sigma^n$, Sp^∞ or L_f are just particular cases.

0. Introduction

Let $\mathcal{S}$ (resp. $\mathcal{S}_*$) be the category of simplicial sets (resp. pointed simplicial sets); we will also denote convenient categories of spaces by these symbols. The base point of $X \in \mathcal{S}_*$ is always denoted by $* \in X$.

0.1. Fibrewise application of functors and Lusternik–Schnirelmann category. Let λ: $\mathcal{S} \to \mathcal{S}$ (or $\mathcal{S}_* \to \mathcal{S}_*$) be a functor together with a natural transformation ι_λ: id $\to \lambda$ as coaugmentation. If λ: $\mathcal{S} \to \mathcal{S}$ is a coaugmented functor and $X \in \mathcal{S}_*$, then $\lambda(X)$ is canonically pointed by $* \to X \to \lambda(X)$, thus λ defines a functor λ': $\mathcal{S}_* \to \mathcal{S}_*$. Throughout this work we suppose that:

- the map $* \to \lambda(*)$ coming from the coaugmentation is a weak equivalence;
- λ preserves weak equivalences.

Such a λ is called a **regular coaugmented functor.**

For any $f \in \mathcal{S}$ there exists a functorial decomposition $f = p_f \circ j_f$ such that j_f is a cofibration and a weak equivalence and p_f a fibration. We fix such a construction and by definition call p_f the **fibration associated** to f. For any point x in the target of f the fibre of p_f over x is called **the homotopy fibre of f over x.** If $f \in \mathcal{S}_*$ the **homotopy fibre of** f indicates the homotopy fibre over $*$.

By [DF96] (see Appendix A) a regular coaugmented functor λ: $\mathcal{S} \to \mathcal{S}$ admits an extension to a functor $\overline{\lambda}$ from the category of spaces over a space to itself such that there are natural transformations

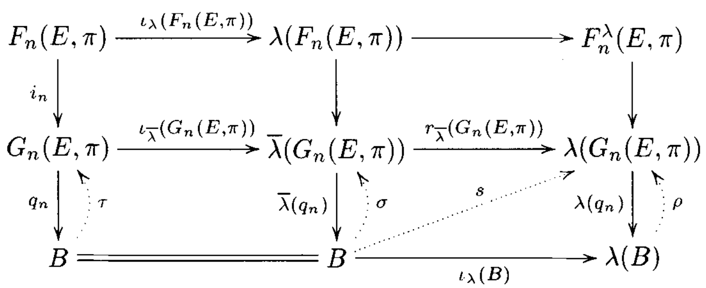
$$\begin{array}{ccccc}
E & \xrightarrow{\iota_{\overline{\lambda}}(E)} & \overline{\lambda}(E) & \xrightarrow{r_{\overline{\lambda}}(E)} & \lambda(E) \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \overline{\lambda}(p)} & & \downarrow{\scriptstyle \lambda(p)} \\
B & = & B & \longrightarrow & \lambda(B)
\end{array}$$

over id_B and $\iota_\lambda(B)$ respectively. Moreover, for p: $E \to B$, the homotopy fibre of $\overline{\lambda}(E) \to B$ over a point x is naturally equivalent to $\lambda(F)$, where F is the homotopy fibre of p over x. We remark that the previous consideration about pointed versions for maps in the image of λ works also with $\overline{\lambda}$.

Applying this construction to the Ganea fibrations we obtain variants of Lusternik–Schnirelmann category. First recall **the Ganea construction** for a map $\pi: E \to B$ in $\mathcal{S}_*$.

Definition 1: Let $q_0: G_0(E,\pi) \to B$ be the fibration associated to π and suppose the fibrations $q_i: G_i(E,\pi) \to B$ have been constructed for $i \leq k-1$. Then we define $q'_k: G_{k-1}(E,\pi) \cup C(F_{k-1}) \to B$ by $q'_{k|G_{k-1}(E,\pi)} := q_{k-1}$ and $q'_{k|C(F_{k-1})} = *$ where $C(F_{k-1})$ is the cone on the fibre F_{k-1} of q_{k-1}. Let $q_k: G_k(E,\pi) \to B$ be the fibration associated to q'_k. In the particular case $\pi = (* \to B)$ we write $q_k: G_k(B) \to B$.

We apply now the fibrewise construction to these fibrations (the dashed arrows correspond to homotopy sections or liftings that are described below):



where $F_n^\lambda(E,\pi)$ is the homotopy fibre of $\lambda(q_n)$. In such a diagram we may consider the existence of a homotopy section τ of q_n, a homotopy section σ of $\overline{\lambda}(q_n)$, a homotopy lifting s of $\iota_\lambda(B)$ through $\lambda(q_n)$ or a homotopy section ρ of $\lambda(q_n)$. The existence of τ is the Ganea definition of the *normalized* LS-category of π, $cat_G(\pi)$, being less than or equal to n. For the others we set:

Definition 2: Let $\lambda: \mathcal{S} \to \mathcal{S}$ be a regular coaugmented functor and $\pi: E \to B$ in $\mathcal{S}_*$. Then:

– **the Ganea λ-category of** π, $\lambda cat_G(\pi)$, is the least integer n (or ∞) such that $\overline{\lambda}(q_n)$ admits a section σ up to pointed homotopy;

– **the Ganea $\lambda_\flat$-category of** π, $\lambda_\flat cat_G(\pi)$, is the least integer n (or ∞) such that there exists $s: B \to \lambda(G_n(E,\pi))$ satisfying $\lambda(q_n) \circ s \simeq \iota_\lambda(B)$;

– **the Toomer λ-invariant of** π, $e_\lambda(\pi)$, is the least integer n (or ∞) such that $\lambda(q_n)$ admits a section ρ up to pointed homotopy.

In the particular case $\pi = (* \to B)$ we write $\lambda cat(B) := \lambda cat_G(* \to B)$, $\lambda_\flat cat(B) := \lambda_\flat cat_G(* \to B)$ and $e_\lambda(B) = e_\lambda(* \to B)$.

As we will see below, this presentation unifies the following approximations of the Lusternik–Schnirelmann category:

– $Mcat$ of a rational space [HL88] is a special case of λcat_G [SS99],

– the strict category weight [Rud99], [Str00], [Van00] fits into the setting of $\lambda_\flat cat_G$,

– the Toomer invariant introduced in [Too74] is equal to e_M for M the abelian group completion of $\lambda = Sp^\infty$, cf. Example 2.

If τ is a section of q_n we get a section of $\overline{\lambda}(q_n)$ by

$$\sigma := \iota_{\overline{\lambda}}(G_n(E,\pi)) \circ \tau.$$

In the same way, if σ is a homotopy section of $\overline{\lambda}(q_n)$ the composite

$$s := r_{\overline{\lambda}}(G_n(E,\pi)) \circ \sigma$$

is a lifting up to homotopy of $\iota_\lambda(B)$ through $\lambda(q_n)$. That is:

$$cat_G(\pi) \geq \lambda cat_G(\pi) \geq \lambda_\flat cat_G(\pi).$$

With an extra hypothesis the existence of a lifting up to homotopy s implies the existence of a homotopical section of $\lambda(q_n)$ (see the end of Section 1):

Proposition 1: *Suppose that λ is a regular coaugmented functor equipped with a natural transformation $\lambda^2 = \lambda \circ \lambda \to \lambda$ whose composition with $\lambda(\iota_\lambda)$ is equal to the identity $\lambda \to \lambda^2 \to \lambda$. Let $s\colon B \to \lambda(G_n(E,\pi))$ such that $\lambda(q_n) \circ s \simeq \iota_\lambda(B)$. Then there exists a homotopical section $\rho\colon \lambda(B) \to \lambda(G_n(E,\pi))$ of $\lambda(q_n)$ and we have $\lambda_\flat cat_G(\pi) = e_\lambda(\pi)$.*

This applies in particular if λ together with the coaugmentation and the transformation $\lambda^2 \to \lambda$ constitutes a triple (see Section 2).

In Definition 2 the subscript $_G$ is chosen to make a distinction from another notion of category of a map due to Fox [Fox41], Berstein and Ganea [BG62] (see also Section 7 of [Jam78]) which admits also variants with a fibrewise construction.

0.2. Unpointed version of pointed functors. In the fibrewise construction $\overline{\lambda}$ associated to a functor $\lambda\colon \mathcal{S} \to \mathcal{S}$ the basepoint free situation is essential and we first meet the problem that the examples of functors that we have in mind, such as the infinite symmetric product, need a basepoint. Therefore for any regular coaugmented functor $\mu\colon \mathcal{S}_* \to \mathcal{S}_*$ we define a canonical functor $\mu^+\colon \mathcal{S} \to \mathcal{S}$ called **basepoint free functor associated to** μ:

For $Y \in \mathcal{S}$ we denote by $Y+ \in \mathcal{S}_*$ the space Y with an extra point added and considered as the basepoint. Let $* \to *+ \to \mu(*+)$ be the map obtained from the canonical inclusion and the coaugmentation. Denote by $*_\mu \to \mu(*+)$

the **fibration associated to the composition** $* \to \mu(*+)$. The functor $Y \mapsto \mu^+(Y)$ is defined by the following pullback:

$$\begin{array}{ccc} \mu^+(Y) & \longrightarrow & \mu(Y+) \\ \downarrow & & \downarrow \\ *_\mu & \longrightarrow & \mu(*+) \end{array}$$

By naturality the composite $Y \to Y+ \to \mu(Y+) \to \mu(*+)$ factorizes as $Y \to * \to \mu(*+)$ and we get a coaugmentation $Y \to \mu^+(Y)$ from the universal property of pullbacks. Note that $\mu^+(Y)$ **is naturally equivalent to the homotopy fibre of** $\mu(Y+) \to \mu(*+)$ **over** $* \in \mu(*+)$.

We will say that a coaugmented functor $\mu\colon \mathcal{S}_* \to \mathcal{S}_*$ **has a basepoint free version** if there exists a coaugmented functor $\lambda\colon \mathcal{S} \to \mathcal{S}$ and a natural transformation between λ' and μ compatible with the coaugmentations and which is a weak equivalence for any $X \in \mathcal{S}_*$. Sometimes, as in Proposition 4 below, μ^+ is a basepoint free version of μ.

Let Σ (resp. Ω) be the reduced suspension (resp. the loop space) in $\mathcal{S}_*$. In this paper we are mainly concerned with the functors $M = Sp^\infty$, $\Omega^n\Sigma^n$, $Q = \lim_\to \Omega^n\Sigma^n$ and their basepoint free functors M^+, $P^n = (\Omega^n\Sigma^n)^+$, Q^+. We also consider the localization functor L_f [DF96]. The functors M and Q are particular cases of a more general construction, the infinite delooping associated to any $\underline{S}$-algebra [Ada78], [EKMM97].

We will see that M^+ (resp. Q^+) is a basepoint free version of M (resp. Q). For $\Omega^n\Sigma^n$ the situation is more complicated: we construct a basepoint free version $Q^n\colon \mathcal{S} \to \mathcal{S}$ which is not homotopically equivalent to $P^n = (\Omega^n\Sigma^n)^+$. We have a general comparison theorem between all these invariants:

THEOREM 1: *Let $\pi\colon E \to B$ in $\mathcal{S}_*$ and $n \geq m$. Then we have the following series of inequalities:*

$$\begin{aligned} cat_G(\pi) \geq & Q^m cat_G(\pi) \geq Q^n cat_G(\pi) \geq P^n cat_G(\pi) \\ \geq & Q cat_G(\pi) \geq M cat_G(\pi) \geq e_M(\pi). \end{aligned}$$

The proof will be given after Example 3 of Section 2.

For rational spaces all the invariants of Theorem 1, except the Toomer invariant, coincide. In fact, in this case, examples for the strict inequality $Mcat(B) > e_M(B)$ can be found in [Fél89, Théorème 12.4.1]. In the last section we will give examples of spaces which show that all the inequalities can be strict except possibly $Q^n cat_G \geq P^n cat_G$. The inequalities in Theorem 1 result from the existence of natural transformations between the related functors.

The functions $P^n cat_G$ and $Qcat_G$ can be compared with stabilized variants of Lusternik–Schnirelmann category studied in [Rud99], [Str00], [Van98], [Van00].

Definition 3: Given $\pi: E \to B$ in $\mathcal{S}_*$, let $\sigma^i cat_G(\pi)$ be the least integer n (or ∞) such that $\Sigma^i G_n(E,\pi) \to \Sigma^i B$ admits a right homotopy inverse. For simplicity we shall write σcat_G for $\sigma^\infty cat_G$.

From the adjunction formula between Ω^i and Σ^i it follows that $\sigma^i cat_G(\pi)= Q^i_\flat cat_G(\pi)$ and therefore, as a particular case of the inequality $\lambda cat_G(\pi) \geq \lambda_\flat cat_G(\pi)$ from above, we obtain:

PROPOSITION 2: *Let $(\pi: E \to B) \in \mathcal{S}_*$. Then one has $Qcat_G(\pi) \geq \sigma cat_G(\pi)$ and $Q^i cat_G(\pi) \geq \sigma^i cat_G(\pi)$.*

0.3. THE INVARIANTS AND CARTESIAN PRODUCTS. It was a question of Ganea [Gan71] called **the Ganea conjecture** whether the equality $cat(Y \times S^r) = cat(Y) + 1$ holds for Y connected and $r \geq 1$. By a result of N. Iwase [Iwa98] this is not always true. It is true, however, that $\sigma cat(Y \times S^r) = \sigma cat(Y) + 1$ by [Rud99], [Van00]. For rational simply connected spaces Y, Z of finite type over the rationals the general formula $cat(Y \times Z) = cat(Y) + cat(Z)$ holds [FHL98] (cf. [Jes90] and [Hes91] for $Z = S^r$).

About our invariants, in particular about $Qcat$, we can state the following:

THEOREM 2: *Let Y, $Z \in \mathcal{S}_*$. Then for $\lambda = Q$, P^n, Q^n, M or for λ a localization functor L_f we have*

$$\lambda cat(Y \times Z) \leq \lambda cat(Y) + \lambda cat(Z).$$

Moreover, the corresponding inequality holds also for $\lambda_\flat cat$.

Remark: The equality $Qcat(X \times S^r) = Qcat(X) + 1$ is true if $Qcat(X) = \sigma cat(X)$. For then

$$Qcat(X \times S^r) \leq Qcat(X) + 1 = \sigma cat(X) + 1 = \sigma cat(X \times S^r)$$

and, by Proposition 2,

$$Qcat(X \times S^r) \geq \sigma cat(X \times S^r) = Qcat(X) + 1.$$

0.4. Hopf invariants. Finally we introduce the notion of Hopf invariants adapted to our situation and prove that they determine if λcat grows when attaching a cell. We also apply them to find examples where the invariants cat, $Q^n cat$, $Qcat$, $Mcat$, σcat are different.

Recall that the counter-examples of Iwase [Iwa98] to the Ganea conjecture are complexes $X = Y \cup_\varphi e^p$ such that the Hopf invariant of φ is not zero but some suspension of it is zero. This phenomenon is, by the definition of $Qcat$, ruled out for the corresponding Hopf invariant. Therefore we may state:

Problem 1: *Does $Qcat$ satisfy the analogue of the Ganea conjecture, i.e., for X connected and $r \geq 1$ does $Qcat(X \times S^r) = Qcat(X) + 1$ hold?*

Indeed, L. Vandembroucq [Van01] answered the question in the positive for finite complexes X.

It follows that for finite connected CW-complexes X with $Qcat(X) = cat(X)$ the Ganea conjecture holds for X. We would like to conjecture that the reverse implication is also true:

Problem 2: *Let X be a finite connected CW-complex. If $Qcat(X) < cat(X)$ then there exists $r \geq 1$ such that $cat(X \times S^r) = cat(X)$.*

In our application of Hopf invariants we verify this for 2-cell complexes. We also construct an infinite CW-complex X such that

$$Qcat(X) < cat(X) \quad \text{and} \quad cat(X \times S^r) = cat(X) + 1 \quad \text{for any } r \geq 1.$$

We may remark also that this example allows a variation of Problem 2 as: *Let X be a connected CW-complex. Then $Q^r cat(X) < cat(X)$ if, and only if,*

$$cat(X \times S^r) = cat(X).$$

We mention finally that, under some restrictions on dimension and connectivity, a mapping version of Problem 2 is proved for rational spaces in [Sta98].

The paper is organized as follows. In Section 1 we recall some more properties of Dror Farjoun's fibrewise application of regular functors. We also study the basepoint free functor associated to a coaugmented functor $\mathcal{S}_* \to \mathcal{S}_*$ and prove Propostion 1. In Section 2 we discuss the case of the functors Q, the abelian group completion M of Sp^∞ and $\Omega^n \Sigma^n$. In fact we defer the topological construction of a basepoint free version of $\Omega^n \Sigma^n$ to Appendix B. In Section 3 we prove Theorem 2 and in Section 4 we present the theory of Hopf invariants for λcat and $\lambda_\flat cat$.

1. Fibrewise application of functors

1.1. Consequences of Dror Farjoun's construction. Let $\lambda\colon \mathcal{S} \to \mathcal{S}$ be a regular coaugmented functor. Let $\pi\colon E \to B$ be a fibration and let $E \to \overline{\lambda}(E)$ over id_B be the construction of Dror Farjoun referred to in the introduction. For the convenience of the reader we will describe it in Appendix A. Directly from it we deduce:

Property 1: *Let $\pi\colon E \to B$ be a fibration with B connected. Let $\lambda_1, \lambda_2\colon \mathcal{S} \to \mathcal{S}$ be two regular coaugmented functors and $\mathcal{L}\colon \lambda_1 \to \lambda_2$ be a natural transformation compatible with the coaugmentations. Then $\mathcal{L}$ induces a natural transformation over B, $\overline{\mathcal{L}}\colon \overline{\lambda}_1 \to \overline{\lambda}_2$. As a consequence we have*

$$\lambda_1 cat_G(\pi) \geq \lambda_2 cat_G(\pi).$$

Moreover, if $\mathcal{L}(Y)$ is a weak equivalence for any $Y \in \mathcal{S}$ then $\overline{\mathcal{L}}(E)$ is a weak equivalence for any E and

$$\lambda_1 cat_G(\pi) = \lambda_2 cat_G(\pi).$$

1.2. Basepoint free version of $\mu\colon \mathcal{S}_* \to \mathcal{S}_*$. We now study the relation between μ and the basepoint free functor $\mu^+\colon \mathcal{S} \to \mathcal{S}$ defined in the introduction. The following two properties are immediate:

Property 2: *Let $\mu_1, \mu_2\colon \mathcal{S}_* \to \mathcal{S}_*$ be two regular coaugmented functors. Let $\mathcal{L}\colon \mu_1 \to \mu_2$ be a natural transformation compatible with the coaugmentations. Then $\mathcal{L}$ induces a natural transformation $\mathcal{L}^+\colon \mu_1^+ \to \mu_2^+$. Moreover, if $\mathcal{L}(X)$ is a weak equivalence for any $X \in \mathcal{S}_*$ then $\mathcal{L}^+(Y)$ is a weak equivalence for any $Y \in \mathcal{S}$.*

In the particular case of the functor $\mu^+\colon \mathcal{S} \to \mathcal{S}$, Proposition 7 of Appendix A implies:

Property 3: *Let $\mu\colon \mathcal{S}_* \to \mathcal{S}_*$ be a regular coaugmented functor and $\mu^+\colon \mathcal{S} \to \mathcal{S}$ the associated basepoint free functor. Let $\pi\colon E \to B$ in $\mathcal{S}_*$ be a fibration with fibre F. Then the homotopy fibre of $\overline{\mu^+}(E) \to B$ is equivalent to the homotopy fibre of $\mu(F+) \to \mu(*+)$ over $*$.*

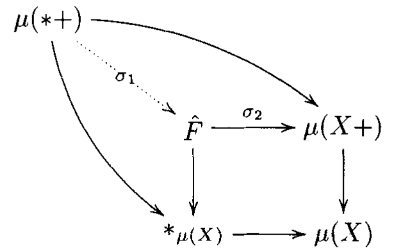
If we start from a basepoint free coaugmented functor $\lambda\colon \mathcal{S} \to \mathcal{S}$, we may compare λ with the associated free construction of the associated functor $\lambda'\colon \mathcal{S}_* \to \mathcal{S}_*$:

PROPOSITION 3: *Let $\lambda: \mathcal{S} \to \mathcal{S}$ be a regular coaugmented functor. Then there exists a natural transformation $\lambda \to (\lambda')^+$ compatible with the coaugmentations.*

We now state a sufficient condition for μ and $(\mu^+)'$ to be equivalent:

Let $\mu: \mathcal{S}_* \to \mathcal{S}_*$ be a regular coaugmented functor with values in the category $\mathcal{G}$ of grouplike spaces (we assume that the base point of a grouplike space is its unit element). For $X \in \mathcal{S}_*$ let $* \to \mu(X)$ be the composition $* \to \mu(*) \to \mu(X)$ and denote by $*_{\mu(X)} \to \mu(X)$ the fibration associated to $* \to \mu(X)$.

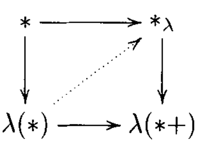
Denote by $X+ \to X$ the canonical map in $\mathcal{S}_*$ (taking $+$ to the basepoint of X) and by $\hat{F}$ the pullback of $*_{\mu(X)} \to \mu(X)$ and $\mu(X+) \to \mu(X)$. The universal property of pullbacks gives a factorization of $\mu(*+) \to \mu(X+)$:



Since the homotopy fibre F of $\mu(X+) \to \mu(X)$ over $*$ is equivalent to $\hat{F}$, we can view σ_1 as a map into F.

PROPOSITION 4: *Using the notation above suppose that $\sigma_1: \mu(*+) \to F$ is a weak equivalence and $\pi_0(\mu(X+)) \to \pi_0(\mu(X))$ is surjective. Then the composite $\mu^+(X) \to \mu(X+) \to \mu(X)$ is a weak equivalence.*

Proof of Proposition 3: In the following square, $* \to \lambda(*)$ is a weak equivalence and a cofibration and $*_\lambda \to \lambda(*+)$ a fibration. Therefore there exists the dashed arrow making the diagram commutative:



The result follows from the definition of $(\lambda')^+$ as a pullback and the existence of a factorization of the composite $\lambda(X) \to \lambda(X+) \to \lambda(*+)$ as $\lambda(X) \to \lambda(*) \to \lambda(*+)$. ∎

Proof of Proposition 4: First we look at the different base points. The universal property of pullbacks gives a factorization of some canonical maps:

$$
\begin{array}{ccccc}
* & & & & \\
 & \searrow & & & \\
 & & \mu^+(X) & \xrightarrow{\;j\;} & \mu(X+) \\
 & & \downarrow & & \downarrow \\
 & & *_\mu & \longrightarrow & \mu(*+)
\end{array}
$$

Therefore $\mu^+(X) \in \mathcal{S}_*$ and $j(*) = *$. Note also that the canonical map $(X+ \to X) \in \mathcal{S}_*$ induces $(\mu(X+) \to \mu(X)) \in \mathcal{G}$, with neutral element $+$ (resp. $*$) in $\mu(X+)$ (resp. $\mu(X)$).

The map $(\mu(X+) \to \mu(*+)) \in \mathcal{G}$ admits a section up to homotopy $\sigma = \sigma_2 \circ \sigma_1$ which gives a homotopy equivalence φ: $\mu(*+) \times \mu^+(X) \to \mu(X+)$, $(\alpha, \beta) \mapsto \sigma_2(\sigma_1(\alpha)).*^{-1}.j(\beta)$. The result follows now from the five lemma applied to the following homotopy commutative diagram of homotopy fibrations:

$$
\begin{array}{ccccc}
F & \xrightarrow{\;\sigma_2\;} & \mu(X+) & \longrightarrow & \mu(X) \\
{\scriptstyle\sigma_1}\uparrow & & \uparrow{\scriptstyle\varphi} & & \uparrow \\
\mu(*+) & \longrightarrow & \mu(*+) \times \mu^+(X) & \longrightarrow & \mu^+(X)
\end{array}
$$

with $\mu(*+) \to \mu(*+) \times \mu^+(X)$, $\alpha \mapsto (\alpha, *)$. ∎

We end this section with the

Proof of Proposition 1: This follows directly from the following diagram

$$
\begin{array}{ccccc}
 & & \lambda^2(G_n(E,\pi))) & \longrightarrow & \lambda(G_n(E,\pi)) \\
 & {\scriptstyle\lambda(s)}\nearrow & \downarrow{\scriptstyle\lambda^2(q_n)} & & \downarrow{\scriptstyle\lambda(q_n)} \\
\lambda(B) & \xrightarrow[\lambda(\iota_\lambda(B))]{} & \lambda^2(B) & \longrightarrow & \lambda(B).
\end{array}
$$

The left triangle homotopy commutes because $\lambda(q_n) \circ s \simeq \iota_\lambda(B)$ and λ, preserving weak equivalences, preserves the homotopy relation. ∎

2. Specific constructions: $\Omega^n\Sigma^n$, $\Omega^\infty\Sigma^\infty$, Sp^∞

Example 1: The functor $Q = \Omega^\infty\Sigma^\infty$ satisfies the assumptions of Proposition 4. In fact let $X \in \mathcal{S}_*$. The homotopy groups of $Q(X)$ constitute a reduced homology theory. From the cofibration sequence $(*+) \to (X+) \to X$ (which admits a retraction $(X+) \to (*+)$) we deduce that the homotopy sequence of $Q(*+) \to Q(X+) \to Q(X)$ decomposes into split short exact sequences. Therefore $Q^+(X) \to Q(X)$ is a homotopy equivalence. We also note that this statement is a particular case of [BE74, Corollary 7.4].

Example 2: Let R be a commutative ring with unit 1. For $X \in \mathcal{S}$ denote by $R \otimes X$ the free R-module generated by X. If $X \in \mathcal{S}_*$ we define $M_R(X) := R \otimes X / R \otimes *$. For $R = \mathbb{Z}$ we obtain in particular $M(X)$. Proposition 4 applies to M_R and $M^+(X)$ is the simplicial set with n-simplices the finite linear combinations $\sum r_i\sigma_i$ of n-simplices of X with $\sum r_i = 1$. This basepoint free version of $M_R(X)$ occurs for example in [BK72]. If $X \in \mathcal{S}_*$ is connected $M(X)$ coincides with the infinite symmetric product $Sp^\infty(X)$.

Remark that Q^+ and M_R are examples of triples.

Example 3: For the construction of a basepoint free version Q^n of $\Omega^n\Sigma^n$: $\mathcal{S}_* \to \mathcal{S}_*$ we refer to Appendix B. We remark that the basepoint free construction $P^n = (\Omega^n\Sigma^n)^+$ is not homotopically equivalent to Q^n.

Proof of Theorem 1: Property 1 is the key point for the comparison between two invariants: we need only to exhibit natural transformations compatible with the coaugmentations.

- There is a natural transformation $Q \to M_{\mathbb{Z}}$ compatible with the coaugmentations (see, e.g., [CM95, 7.3]). An easy way to construct it is to use the combinatorial model of Barratt and Eccles [BE74]. Thus we have $Qcat_G(\pi) \geq M_{\mathbb{Z}}cat_G(\pi)$.
- If $m \leq n$ the natural transformation $Q^m \to Q^n$ in *Top* gives $Q^m cat_G(\pi) \geq Q^n cat_G(\pi)$; cf. Appendix B.
- From Proposition 3 we get a natural transformation $Q^n \to ((Q^n)')^+$. By Appendix B there is a natural transformation $(Q^n)' \to \Omega^n\Sigma^n$; thus composing $Q^n \to ((Q^n)')^+ \to (\Omega^n\Sigma^n)^+$ provides a natural transformation $Q^n \to P^n$.
- The inequality $P^n cat_G(\pi) \geq Qcat_G(\pi)$ comes from the natural transformation $\Omega^n\Sigma^n \to \Omega^\infty\Sigma^\infty$.
- The existence of a homotopical section to $M(q_n)$ could be chosen as a definition for Toomer's invariant. Therefore $Mcat(X) \geq e_M(X)$ is a direct consequence of Proposition 1.

- The remaining inequality $cat_G(\pi) \geq Q^n cat_G(\pi)$ is obvious. ∎

3. Product formulae

The proof of Theorem 2 will be based on the following result of [SS99]: Let $\lambda: \mathcal{S} \to \mathcal{S}$ be a regular coaugmented functor. If there is a natural transformation $\lambda(Y) \times \lambda(Z) \to \lambda(Y \times Z)$ which is compatible with the coaugmentations, then the product formula $\lambda cat(Y \times Z) \leq \lambda cat(Y) + \lambda cat(Z)$ holds. The corresponding formula for $\lambda_\flat cat$ follows even more easily.

PROPOSITION 5: *Suppose that $\lambda: \mathcal{S} \to \mathcal{S}$ is coaugmented by $\iota_\lambda: id \to \lambda$. Assume that there are natural transformations $\lambda(Y) \times Z \to \lambda(Y \times Z)$ and $m: \lambda^2 \to \lambda$ which are compatible with the coaugmentations.*

Then there is a natural transformation $\lambda(Y) \times \lambda(Z) \to \lambda(Y \times Z)$.

Proof: The transformation consists of the composition $\lambda(Y) \times \lambda(Z) \to \lambda(Y \times \lambda(Z)) \to \lambda^2(Y \times Z) \to \lambda(Y \times Z)$. ∎

Remark: If λ is as in Proposition 1, then there exists m with $m \circ \iota_\lambda(d(Y)) = id_{\lambda(Y)}$ for $Y \in \mathcal{S}$.

COROLLARY 1: *Let $\mu: \mathcal{S}_* \to \mathcal{S}_*$ be coaugmented such that there exist natural transformations $\mu(X) \times X' \to \mu(X \times X')$ and $\mu^2(X) \to \mu(X)$ compatible with the coaugmentations.*

Then there is a natural transformation $\mu^+(Y) \times \mu^+(Z) \to \mu^+(Y \times Z)$ of functors $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$.

Proof: Let $\overline{\mu}(Y) = \mu(Y+)$ for $Y \in \mathcal{S}$. Then it suffices to show that $\overline{\mu}$ satisfies the assumptions on λ of Proposition 5.

(a) $\overline{\mu}(Y) \times Z = \mu(Y+) \times Z \to \mu(Y+) \times (Z+) \to \mu((Y+) \times (Z+)) \to \mu((Y \times Z)+) = \overline{\mu}(Y \times Z)$. The last arrow is induced by the canonical map $(Y+) \times (Z+) \to (Y \times Z)+$.

(b) $\overline{\mu}^2(Y) = \mu((\mu(Y+))+) \to \mu(\mu(Y+)) \to \mu(Y+) = \overline{\mu}(Y)$. The first arrow is induced by the map $(\mu(Y+))+ \to \mu(Y+)$ which is the identity on $\mu(Y+)$ and maps $+$ to the basepoint $+ \in \mu(Y+)$. ∎

Proof of Theorem 2: We need only to observe from Proposition 8 of Appendix B that the basepoint free versions of Q and $\Omega^n\Sigma^n$ satisfy the assumptions of Proposition 5.

The combinatorial models Γ for $\Omega^\infty\Sigma^\infty$ of [BE74] and Γ^n for $\Omega^n\Sigma^n$ of [Smi89] are convenient too. It has been shown directly in [BE74] that Γ in particular satisfies the assumptions of Corollary 1. A close look at the combinatorial details shows that this is also true for $\Gamma^n \subset \Gamma$. Thus the functor P^n satisfies the conditions of Proposition 5.

For $\mu = \Omega^n\Sigma^n$ we can also argue topologically. The second transformation needed in Corollary 1 exists for μ but —perhaps— not the first one. However, we show that $\overline{\mu}$ admits a natural transformation $\overline{\mu}(Y) \times Z \to \overline{\mu}(Y \times Z)$ compatible with the coaugmentations. It follows that μ^+ (hence P^n) is a functor as in Proposition 5.

To give the required formula we write $\Sigma^n(Y+) = S^n \wedge (Y+) = S^n \rtimes Y$ where $\rtimes\colon \mathcal{S}_* \times \mathcal{S} \to \mathcal{S}_*$ is the halfsmash. Then we have

$$\Sigma^n((Y \times Z)+) = S^n \rtimes (Y \times Z) = (S^n \rtimes Y) \rtimes Z.$$

We define $\Phi\colon \Omega^n\Sigma^n(Y+) \times Z \to \Omega^n\Sigma^n((Y \times Z)+)$ by $\Phi(w,z)(t) = [w(t), z]$ where $w\colon S^n \to \Sigma^n(Y+)$, $t \in S^n$, and $[w(t), z]$ denotes the class of $(w(t), z)$ in $(S^n \rtimes Y) \rtimes Z$.

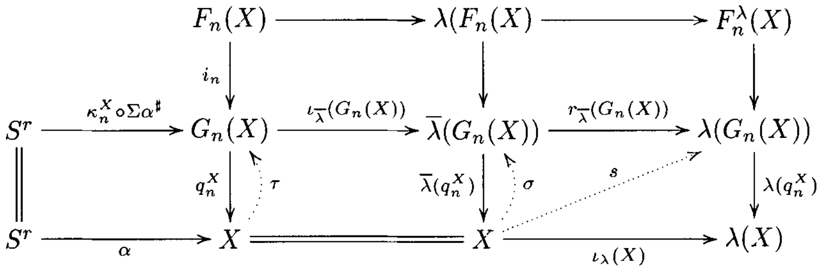
For the localization functor, L_f, we observe [DF96, pages 21–23] the existence of a natural transformation $L_f(Y) \times Z \to L_f(Y \times Z)$ which gives a natural transformation $L_f(Y) \times L_f(Z) \to L_fL_f(Y \times Z)$. The coaugmentation induces a weak equivalence $L_f \to L_fL_f$ and we deduce $L_fcat(Y \times Z) = L_fL_fcat(Y \times Z) \leq L_fcat(Y) + L_fcat(Z)$. ∎

4. Hopf invariants

Let $X \in \mathcal{S}_*$ and $\alpha\colon S^r \to X$ be a map with cofibre $Y = X \cup_\alpha e^{r+1}$.

We will characterize the relationship between the different λ LS-type invariants of X and Y in terms of a homotopy class associated to α and called a **Hopf invariant**. We use the presentation of [Iwa98]. For $\lambda = id$ this coincides with the Berstein–Hilton definition [BH60] (see [Van98, Proposition 3.2.7] for a detailed proof). In this section we will make no distinction between maps and (pointed) homotopy classes of maps.

4.1. Definition and Properties. Consider first the adjoint $\alpha^\sharp\colon S^{r-1} \to \Omega X$ of α whose supension gives a homotopy class $\Sigma\alpha^\sharp\colon S^r \to \Sigma\Omega X$ into the first Ganea space associated to X. By composition with the maps $\kappa_n^X\colon \Sigma\Omega X \to G_n(X)$ coming from the construction of the Ganea fibrations we have maps $\kappa_n^X \circ \Sigma\alpha^\sharp\colon S^r \to G_n(X)$. We work with the absolute case and the situation described in the introduction becomes:



Recall that $\iota_\lambda(G_n(X)) = r_{\overline{\lambda}}(G_n(X)) \circ \iota_{\overline{\lambda}}(G_n(X))$.

Definition 4: (1) Suppose that $\overline{\lambda}(q_n^X)\colon \overline{\lambda}(G_n(X)) \to X$ admits a homotopical section σ. Then **the Hopf-invariant associated to** $(\sigma, \lambda, \alpha)$ is:

$$\mathcal{H}'_{\sigma,\lambda}(\alpha) := \left(\iota_{\overline{\lambda}}(G_n(X)) \circ \kappa_n^X \circ \Sigma\alpha^\sharp\right) - (\sigma \circ \alpha) \in \pi_r(\overline{\lambda}(G_n(X))).$$

(2) Suppose there exists $s\colon X \to \lambda(G_n(X))$ such that $\lambda(q_n^X) \circ s \simeq \iota_\lambda(X)$. Then **the Hopf-invariant associated to** (s, λ, α) is:

$$H_{s,\lambda}(\alpha) := \left(\iota_\lambda(G_n(X)) \circ \kappa_n^X \circ \Sigma\alpha^\sharp\right) - (s \circ \alpha) \in \pi_r(\lambda(G_n(X))).$$

Remark: Consider $\beta\colon S^t \to S^r$ a coH-map (for instance, a suspension) and $\alpha\colon S^r \to X$. Directly from Definition 4 we have $\mathcal{H}_{\sigma,\lambda}(\alpha \circ \beta) = \mathcal{H}_{\sigma,\lambda}(\alpha) \circ \beta$ and $H_{\sigma,\lambda}(\alpha \circ \beta) = H_{\sigma,\lambda}(\alpha) \circ \beta$.

The element $\mathcal{H}'_{\sigma,\lambda}(\alpha) \in \pi_r(\overline{\lambda}(G_n(X)))$ lifts in the fibre as an element denoted by $\mathcal{H}_{\sigma,\lambda}(\alpha) \in \pi_r(\lambda(F_n(X)))$ and there is no indeterminacy in this lifting because $\lambda(F_n(X)) \to \overline{\lambda}(G_n(X))$ induces an injection between homotopy groups. Notice that we are distinguishing between $\mathcal{H}_{\sigma,\lambda}$ and $\mathcal{H}'_{\sigma,\lambda}$. We do this because though $\mathcal{H}'_{\sigma,\lambda}$ always determines $\mathcal{H}_{\sigma,\lambda}$, $\iota_\lambda(G_n(X))_* \circ \mathcal{H}'_{\sigma,\mathrm{id}}$ does not determine $\iota_\lambda(G_n(X))_* \circ \mathcal{H}_{\sigma,\mathrm{id}}$. This turns out to be one source of examples where the invariants we study differ; cf. Corollary 2.

We consider the classical Hopf invariant of Berstein–Hilton [BH60] as a particular case of $\mathcal{H}_{\sigma,\lambda}$ for $\lambda = id$ and use, in this case, the notation $\mathcal{H}_\sigma$ (or H_σ). If there is a unique homotopy class of section we shorten the notation in $\mathcal{H}$ (or H).

The LS-category of the skeleton of a non-contractible CW-complex is always less than or equal to the LS-category of the total space [Sta00]. This property can be extended to the setting of λcat as follows:

THEOREM 3: *Let $\lambda: \mathcal{S} \to \mathcal{S}$ be a regular coaugmented functor preserving k-equivalences for any $k > 0$. Let X be a $(k-1)$-connected CW-complex and $X^{(r)}$ be its r-skeleton. We suppose $r \geq k$, ($k \geq 2$ and $n \geq 1$) or ($k = 1$ and $n \geq 2$).*

For any section σ of $\overline{\lambda}(q_n^X)$: $\overline{\lambda}(G_n(X)) \to X$, $n \geq 1$, there exists a compatible section σ_r of $\overline{\lambda}(G_n(X^{(r)})) \to X^{(r)}$. In other words the following diagram commutes:

$$\begin{array}{ccc} \overline{\lambda}(G_n(X^{(r)})) & \longrightarrow & \overline{\lambda}(G_n(X)) \\ \downarrow \uparrow \sigma_r & & \downarrow \uparrow \sigma \\ X^{(r)} & \longrightarrow & X \end{array}$$

As a consequence, if X is simply connected and $cat(X) \geq 1$ or X is connected and $cat(X) \geq 2$, we have $\lambda cat(X^{(r)}) \leq \lambda cat(X)$, for any $r \geq k$.

We show now that the Hopf invariant characterizes in a certain way the growth of the LS-category when a cell is attached to a CW-complex. The following theorem generalizes results of [BH60], [Iwa98], [Sta00] and [Van98]:

THEOREM 4: *Let $\lambda: \mathcal{S} \to \mathcal{S}$ be a regular coaugmented functor and X be a connected space of associated Ganea fibration q_n^X: $G_n(X) \to X$. Consider $\alpha: S^r \to X$. Denote by $Y = X \cup_\alpha e^{r+1}$ the space X with a cell attached along α and by $\rho: X \to Y$ the canonical inclusion.*

(1) If there is some homotopy section σ of $\overline{\lambda}(q_n^X)$ such that $\mathcal{H}_{\sigma,\lambda}(\alpha) = 0$ then $\lambda cat(Y) \leq n$.

(2) We suppose $n > 1$ or X simply connected. If λ preserves $(r+1)$-equivalences, $r > 1$ and $\dim X \leq r$ then: $\lambda cat(Y) \leq n$ iff there exists a homotopy section σ of $\overline{\lambda}(q_n^X)$ such that $\mathcal{H}_{\sigma,\lambda}(\alpha) = 0$.

(3) Suppose that λ is a regular coaugmented functor equipped with a natural transformation $\lambda^2 = \lambda \circ \lambda \to \lambda$ whose composition with $\lambda(\iota_\lambda)$ is equal to the identity $\lambda \to \lambda^2 \to \lambda$. If there exists $s: X \to \lambda(G_n(X))$ such that $\lambda(q_n^X) \circ s \simeq \iota_\lambda(X)$ and $H_{s,\lambda}(\alpha) = 0$ then $\lambda_\flat cat(Y) \leq n$.

The hypothesis required on λ in the statements (2) and (3) are satisfied by the functors Q^n, P^n, Q, $M = Sp^\infty$.

Suppose there exists a natural transformation $\mathcal{L}: \lambda_1 \to \lambda_2$ compatible with the coaugmentations between two regular coaugmented functors. If σ_1 is a homotopical section of $\overline{\lambda}_1(q_n^X)$ we define a homotopical section of $\overline{\lambda}_2(q_n^X)$ by $\sigma_2 := \mathcal{L}(G_n(X)) \circ \sigma_1$ and we have $\mathcal{H}'_{\sigma_2,\lambda_2} = \mathcal{L}(G_n(X)) \circ \mathcal{H}'_{\sigma_1,\lambda_1}$. We may also define a lifting s_1 from σ_1 and the Hopf invariant H_{s_1,λ_1} is obtained from $\mathcal{H}'_{\sigma_1,\lambda_1}$

by composition with $\overline{\lambda}_1(G_n(X)) \to \lambda_1(G_n(X))$. These considerations and Theorem 4 give us directly a relationship between the different Hopf invariants associated to our functors:

COROLLARY 2: *Let X be a simply connected space of LS-category n with a section $\tau: X \to G_n(X)$ to the Ganea fibration q_n^X. Let $\alpha: S^r \to X$ and $Y = X \cup_\alpha e^{r+1}$. Denote by $\mathcal{H}_\tau(\alpha) \in \pi_r(F_n(X))$ and $\mathcal{H}'_\tau(\alpha) \in \pi_r(G_n(X))$ the Hopf invariants associated to (τ, α) and by Hur the Hurewicz homomorphism. Then we have:*

- $\Sigma^i \mathcal{H}_\tau(\alpha) = 0 \Rightarrow Q^i cat(Y) \leq n$;
- $\Sigma^i \mathcal{H}'_\tau(\alpha) = 0 \Rightarrow \sigma^i cat(Y) \leq n$;
- $\mathrm{Hur}\, \mathcal{H}_\tau(\alpha) = 0 \Rightarrow Mcat(Y) \leq n$;
- $\mathrm{Hur}\, \mathcal{H}'_\tau(\alpha) = 0 \Rightarrow e(Y) \leq n$.

Coming back to the general situation we will prove that Theorem 4 implies:

COROLLARY 3: *Let λ be a regular coaugmented functor and $\alpha: S^r \to X$. Then*

$$\lambda cat(X \cup_\alpha e^{r+1}) \leq \lambda cat(X) + 1.$$

The argument used in the proof of Corollary 3 does not work for $\lambda_\flat cat$. In fact, by [KV00], there is an example X with $e(X \cup_\alpha e^{r+1}) = e(X) + 2$.

We present now some particular results used in the proofs:

LEMMA 1: *Consider the situation of Theorem 4 and let $\overline{\lambda}(G_n(\rho)): \overline{\lambda}(G_n(X)) \to \overline{\lambda}(G_n(Y))$ and $\lambda(G_n(\rho)): \lambda(G_n(X)) \to \lambda(G_n(Y))$ be the maps induced by $\rho: X \to Y$.*

(i) *If $\overline{\lambda}(q_n^X)$ admits a homotopy section σ we have:*

$$\overline{\lambda}(G_n(\rho)) \circ \sigma \circ \alpha \simeq -\left(\overline{\lambda}(G_n(\rho)) \circ \mathcal{H}'_{\sigma,\lambda}(\alpha)\right).$$

(ii) *If s exists we have:*

$$\lambda(G_n(\rho)) \circ s \circ \alpha \simeq -\left(\lambda(G_n(\rho)) \circ H_{s,\lambda}(\alpha)\right).$$

LEMMA 2: *Let $r \geq k \geq 1$. Let B be a $(k-1)$-connected CW-complex of dimension $\leq r$. Consider the cofibration $\vee_J S^r \to B \to C = B \cup_J e^{r+1}$. Let $k \geq 2$ or ($k = 1$ and $n \geq 2$). Then, for $n \geq 1$, the map $B \to C$ induces an $(r+1)$-equivalence $F_n(B) \to F_n(C)$ between the fibres of Ganea fibrations.*

LEMMA 3: *Let* $S^r \xrightarrow{\alpha} B \xrightarrow{\rho} C = B \cup_\alpha e^{r+1}$ *be a cofibration and* $p\colon Y \to C$ *be a map such that* $\pi_{r+1}(p)$ *is surjective. Let* $\varphi\colon B \to Y$ *be a map such that* $\varphi \circ \alpha \simeq *$ *and* $p \circ \varphi \simeq \rho$*. Then there exists* $\sigma\colon C \to Y$ *such that* $\sigma \circ \rho \simeq \varphi$ *and* $p \circ \sigma \simeq id_C$.

The end of this section is devoted to proofs beginning with the proofs of the Lemmas.

Proof of Lemma 1: By definition we have:

$$\overline{\lambda}(G_n(\rho)) \circ \sigma \circ \alpha = \overline{\lambda}(G_n(\rho)) \circ [-(\mathcal{H}'_{\sigma,\lambda}(\alpha)) + \iota_{\overline{\lambda}}(G_n(X)) \circ \kappa_n^X \circ \Sigma\alpha^\sharp].$$

The required equality follows from

$$\overline{\lambda}(G_n(\rho)) \circ \iota_{\overline{\lambda}}(G_n(X)) \circ \kappa_n^X \circ \Sigma\alpha^\sharp \simeq \iota_{\overline{\lambda}}(G_n(Y)) \circ \kappa_n^Y \circ \Sigma\Omega\rho \circ \Sigma\alpha^\sharp \simeq *.$$

The verification of (ii) is similar. ∎

Proof of Lemma 2: Observe that the fibre $F_n(B)$ (resp. $F_n(C)$) having the homotopy type of the iterated join $*^{n+1}\Omega B$ (resp. $*^{n+1}\Omega C$) implies that it is $((n+1)k-2)$-connected. With the assumptions on k and n, $F_n(B)$ and $F_n(C)$ are simply connected. A homology argument shows that the induced map $F_n(B) \to F_n(C)$ is an $(nk+r-1)$-equivalence and thus an $(r+1)$-equivalence. ∎

Proof of Lemma 3: The map p induces a morphism between the following two long exact sequences coming from the cofibration $S^r \to B \to C$:

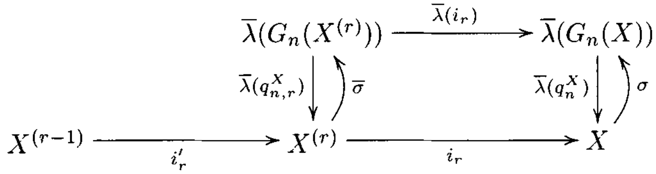
$$\begin{array}{ccccccc} \longrightarrow [S^{r+1}, Y] & \longrightarrow & [C, Y] & \longrightarrow & [B, Y] & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow [S^{r+1}, C] & \longrightarrow & [C, C] & \longrightarrow & [B, C] & \longrightarrow \end{array}$$

From $\varphi \circ \alpha \simeq *$ we deduce the existence of $\psi\colon C \to Y$ such that $\psi \circ \rho \simeq \varphi$. The elements $p \circ \psi$ and id_C of $[C, C]$ satisfy $p \circ \psi \circ \rho \simeq id \circ \rho$. By a theorem of D. Puppe [Hil67, Theorem 15.4] there exists $\xi' \in [S^{r+1}, C]$ such that $(p \circ \psi)^{\xi'} \simeq id_C$ where $(p \circ \psi)^{\xi'}$ denotes the cooperation of ξ' on $p \circ \psi$ induced by the cofibration.

By hypothesis there exists $\xi \in [S^{r+1}, Y]$ such that $\xi' \simeq p \circ \xi$. Set $\sigma = \psi^\xi$. Then we have $p \circ \sigma = p \circ (\psi)^\xi \simeq (p \circ \psi)^{p \circ \xi} \simeq id_C$. ∎

Proof of Theorem 3: Denote by $i_r\colon X^{(r)} \to X$ and $i'_r\colon X^{(r-1)} \to X^{(r)}$ the canonical inclusions and by $q^X_{n,r}\colon G_n(X^{(r)}) \to X^{(r)}$ the Ganea fibration. Let σ

be any section of $\overline{\lambda}(q_n^X)$. The map i_r induces a morphism of fibrations between $\overline{\lambda}(q_{n,r}^X)$ and $\overline{\lambda}(q_n^X)$ which is an r-equivalence between the bases and an $(r+1)$-equivalence between the fibres (by Lemma 2 and the hypothesis on λ). Also the Ganea fibrations split after looping. So with the homotopy long exact sequences we deduce that $\overline{\lambda}(i_r)$ is an r-equivalence. Therefore there exists $\overline{\sigma}$ such that in the following diagram



$\overline{\lambda}(i_r)\circ\overline{\sigma}\simeq\sigma\circ i_r$ and $\overline{\lambda}(q_{n,r}^X)\circ\overline{\sigma}\circ i'_r\simeq i'_r$. By Lemma 3 applied to the cofibration $\vee S^{r-1}\to X^{(r-1)}\to X^{(r)}$ we can choose an element $\xi'\in[\vee S^r, X^{(r)}]$ such that $(\overline{\lambda}(q_{n,r}^X)\circ\overline{\sigma})^{\xi'}\simeq id$. Now, $\pi_r(\overline{\lambda}(q_{n,r}^X))$ being surjective, we can choose $\xi\in[\vee S^r,\overline{\lambda}(G_n(X^{(r)}))]$ with $\pi_r(\overline{\lambda}(q_{n,r}^X))(\xi)=\xi'$. Hence,

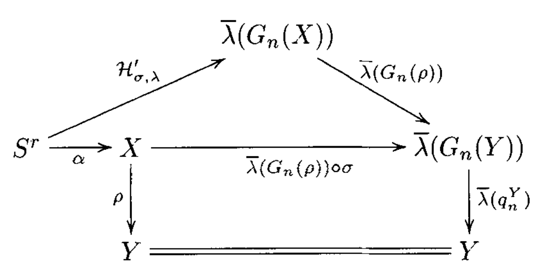
$$\overline{\lambda}(q_{n,r}^X)\circ\overline{\sigma}^{\xi}\simeq(\overline{\lambda}(q_{n,r}^X)\circ\overline{\sigma})^{\xi'}\simeq id.$$

We may homotope $\overline{\sigma}^{\xi}$ to a section σ'_r of $\overline{\lambda}(q_{n,r}^X)$ such that $\overline{\lambda}(i_r)\circ\sigma'_r\circ i'_r=\sigma\circ i_r\circ i'_r$. We can therefore find $\eta''\in[\vee S^r,\overline{\lambda}(G_n(X))]$ with $(\overline{\lambda}(i_r)\circ\sigma'_r)^{\eta''}\simeq\sigma\circ i_r$. From

$$\overline{\lambda}(q_n^X)\circ\overline{\lambda}(i_r)\circ\sigma'_r\simeq\overline{\lambda}(q_n^X)\circ\sigma\circ i_r$$

we deduce that $\overline{\lambda}(q_n^X)\circ\eta''$ acts trivially on $\overline{\lambda}(q_n^X)\circ\overline{\lambda}(i_r)\circ\sigma'_r$. Then the element $\eta':=\eta''-\sigma\circ\overline{\lambda}(q_n^X)\circ\eta''\in[\vee S^r,\lambda(F_n(X))]$ satisfies $\left(\overline{\lambda}(i_r)\circ\sigma'_r\right)^{\eta'}\simeq\left(\overline{\lambda}(i_r)\circ\sigma'_r\right)^{\eta''}\simeq\sigma\circ i_r$. Let $\eta\in[\vee S^r,\lambda(F_n(X^{(r)}))]$ be an element which is mapped to η' by the map induced by $\lambda(F_n(X^{(r)}))\to\lambda(F_n(X))$; set $\sigma_r:=(\sigma'_r)^{\eta}$. Then σ_r is still a section of $\overline{\lambda}(q_{n,r}^X)$ with $\overline{\lambda}(i_r)\circ\sigma_r\simeq\sigma\circ i_r$. ∎

Proof of Theorem 4: Suppose that $\overline{\lambda}(q_n^X)$ admits a section σ. By application of Lemma 1 (i) we get a commutative diagram (up to sign):



1) If $\mathcal{H}'_{\sigma,\lambda} \simeq *$ we apply Lemma 3 to construct a map $\sigma'\colon Y \to \overline{\lambda}(G_n(Y))$ such that $\overline{\lambda}(G_n(\rho)) \circ \sigma \simeq \sigma' \circ \rho$ and $\overline{\lambda}(q_n^Y) \circ \sigma' \simeq id_Y$. By definition we have $\lambda cat(Y) \leq n$.

2) Let $\sigma'\colon Y \to \overline{\lambda}(G_n(Y))$ be a section of $\overline{\lambda}(q_n^Y)$. By Theorem 3 there exists a section σ of $\overline{\lambda}(q_n^X)$ such that $\sigma' \circ \rho \simeq \overline{\lambda}(G_n(\rho)) \circ \sigma$. From the diagram above we deduce immediately that $\overline{\lambda}(G_n(\rho)) \circ \mathcal{H}'_{\sigma,\lambda} \simeq *$. This implies that $\lambda(F_n(\rho)) \circ \mathcal{H}_{\sigma,\lambda} \simeq *$ by injectivity of $\pi_r(\lambda(F_n(Y))) \to \pi_r(\overline{\lambda}(G_n(Y)))$ and that $\mathcal{H}_{\sigma,\lambda} \simeq *$ by Lemma 2 and the hypothesis on λ.

3) Set $\tilde{\alpha} := \iota_\lambda(X) \circ \alpha\colon S^r \to \lambda(X)$ and $\overline{\alpha} := s \circ \alpha\colon S^r \to \lambda(G_n(X))$. Note that $\lambda(q_n^X) \circ \overline{\alpha} = \tilde{\alpha}$ and, because of $H_{s,\lambda}(\alpha) = 0$, $\overline{\alpha} \simeq \iota_\lambda(G_n(X)) \circ \kappa_n \circ \Sigma\alpha^\sharp$. From naturality of ι_λ we have $\lambda(\rho) \circ \tilde{\alpha} \simeq *$ and we deduce from Lemma 1 (ii) that $\lambda(G_n(\rho)) \circ \overline{\alpha} \simeq *$. The universal property of pushouts and, for the right bottom square, [Van00, Proposition 2.5] give a homotopy commutative diagram (without the dashed arrow):

$$\begin{array}{ccccccc}
S^r & \xrightarrow{\alpha} & X & \longrightarrow & X \cup_\alpha e^{r+1} & = & Y \\
\| & & \downarrow{\scriptstyle \iota_\lambda(X)} & & \downarrow & & \downarrow{\scriptstyle \iota_\lambda(Y)} \\
S^r & \xrightarrow{\tilde{\alpha}} & \lambda(X) & \longrightarrow & \lambda(X) \cup_{\tilde{\alpha}} e^{r+1} & \longrightarrow & \lambda(Y) \\
\| & & \uparrow{\scriptstyle \lambda(q_n^X)} & & {\scriptstyle \tilde{q}_n}\uparrow \; \vdots{\scriptstyle \tilde{s}_n} & & \uparrow{\scriptstyle \lambda(q_n^Y)} \\
S^r & \xrightarrow[\overline{\alpha}]{} & \lambda(G_n(X)) & \longrightarrow & \lambda(G_n(X)) \cup_{\overline{\alpha}} e^{r+1} & \longrightarrow & \lambda(G_n(Y))
\end{array}$$

where $\lambda(G_n(X)) \to \lambda(G_n(X)) \cup_{\overline{\alpha}} e^{r+1} \to \lambda(G_n(Y))$ is homotopic to $\lambda(G_n(\rho))$ and $\lambda(X) \to \lambda(X) \cup_{\tilde{\alpha}} e^{r+1} \to \lambda(Y)$ is homotopic to $\lambda(\rho)$.

From the hypothesis on λ and Proposition 1 one has a homotopical section $\overline{s}_n$ to $\lambda(q_n^X)$; a look at its construction gives $\overline{s}_n \circ \tilde{\alpha} \simeq \overline{\alpha}$. Denote by $\overline{\overline{s}}_n$ and $\tilde{q}_n$ the maps induced by $\overline{s}_n$ and $\lambda(q_n^X)$ between the cofibres. The map $\varphi = \tilde{q}_n \circ \overline{\overline{s}}_n$ induced by $\lambda(q_n^X) \circ \overline{s}_n \simeq id$ is a homotopy equivalence [Qui67, Section I.3]. By composing $\overline{\overline{s}}_n$ with φ^{-1} we get a homotopical section $\tilde{s}_n$ of $\tilde{q}_n$. The required homotopy lifting of $Y \to \lambda(Y)$ through $\lambda(q_n^Y)$ is the following composite:

$$X \cup_\alpha e^{r+1} \longrightarrow \lambda(X) \cup_{\tilde{\alpha}} e^{r+1} \xrightarrow[\tilde{s}_n]{} \lambda(G_n(X)) \cup_{\overline{\alpha}} e^{r+1} \longrightarrow \lambda(G_n(Y)). \qquad \blacksquare$$

Proof of Corollary 3: The triviality of the induced map $F_n(Y) \to F_{n+1}(Y)$ implies the triviality of $\lambda(F_n(Y)) \to \lambda(F_{n+1}(Y))$ and the image of the Hopf invariant $\mathcal{H}_{\sigma,\lambda}(\alpha)$ in $\pi_*(\lambda(F_{n+1}(Y)))$ is zero. As in the beginning of the proof of

Theorem 4 we construct a dashed arrow making commutative

$$\begin{array}{ccc} X & \longrightarrow & \overline{\lambda}(G_{n+1}(Y)) \\ \big\downarrow & \nearrow & \big\downarrow \lambda(q^Y_{n+1}) \\ Y & = & Y \end{array}$$

In other words $\lambda cat(Y) \leq n+1$. ∎

4.2. Examples. We come back to the chain of inequalities of Theorem 1 and exhibit examples of spaces for which a strict inequality occurs (except for P^n and Q^n). For this we will apply Corollary 2.

Example 4: We use the notation and results of [Tod62, Proposition 13.9 page 179]. The composite $\beta := \alpha_1(3) \circ \alpha_1(2p)$: $S^{4p-3} \to S^{2p} \to S^3$ is a generator of $\pi_{4p-3}(S^3) = \mathbb{Z}_p$ such that $\Sigma\beta \not\simeq *$ and $\Sigma^2\beta \simeq *$. Denote by w: $S^4 \to S^3 \vee S^2$ the Whitehead bracket of the classes S^3 and S^2 and by $\gamma := w \circ \Sigma\beta$: $S^{4p-2} \to S^4 \to S^3 \vee S^2$. Set $X = \left(S^3 \vee S^2\right) \cup_\gamma e^{4p-1}$. Then we claim $Q^1cat(X) = 1$ and $cat(X) = 2$ (cf. also [Sta98] for a different proof of $cat(X) = 2$).

The Hopf invariant of γ satisfies $\mathcal{H}(\gamma) = \mathcal{H}(w \circ \Sigma\beta) = \mathcal{H}(w) \circ \Sigma\beta$. Therefore $\Sigma\mathcal{H}(\gamma) \simeq *$ and $Q^1cat(X) = 1$ by Corollary 2. We are now reduced to proving that $\mathcal{H}(\gamma)$ is not trivial. Denote by $f^\sharp$ the adjoint of a map f and observe that $\mathcal{H}(\gamma)^\sharp = \mathcal{H}(w)^\sharp \circ \beta$. The non-triviality of $\mathcal{H}(\gamma)$ is a consequence of the following lemma. It is certainly well known but we cannot find it in the literature.

Lemma 4: *Let $i, j \geq 2$. Let $w_{i,j}$: $S^{i+j-1} \to S^i \vee S^j$ be the Whitehead bracket of the canonical inclusions η^i: $S^i \hookrightarrow S^i \vee S^j$, η^j: $S^j \hookrightarrow S^i \vee S^j$. Denote by $F_{i,j}$ the homotopy fibre of the first Ganea fibration associated to $S^i \vee S^j$. The Hopf invariant associated to $w_{i,j}$ has for adjoint a map $\mathcal{H}(w_{i,j})^\sharp$: $S^{i+j-2} \to \Omega F_{i,j}$.*

Then there exists a map $\overline{p}$: $\Omega F_{i,j} \to \Omega S^{i+j-1}$ such that the adjoint of $\overline{p} \circ \mathcal{H}(w_{i,j})^\sharp$ is a map of degree ± 1: $S^{i+j-1} \to S^{i+j-1}$.

Proof: By the Hilton–Milnor theorem [Whi78, page 515]:

$$\Omega(S^i \vee S^j) \simeq \Omega S^i \times \Omega S^j \times \Omega S^{i+j-1} \times \cdots.$$

Recall that $w^\sharp_{i,j}$ is constructed using the commutator of $S^{i-1} \to \Omega S^i \to \Omega(S^i \vee S^j)$ and $S^{j-1} \to \Omega S^j \to \Omega(S^i \vee S^j)$; the extension of $w^\sharp_{i,j}$: $S^{i+j-2} \to \Omega(S^i \vee S^j)$ to $\Omega\Sigma S^{i+j-2}$ is the inclusion $\Omega S^{i+j-1} \to \Omega(S^i \vee S^j)$ in the above decomposition. Note that there is one homotopy section of $\Sigma\Omega(S^i \vee S^j) \to S^i \vee S^j$ up to homotopy.

It follows that there is a map $\overline{p}\colon \Omega F_{i,j} \to \Omega S^{i+j-1}$ with the adjoint of $\overline{p}\circ\mathcal{H}(w_{i,j})^\sharp$ a map of degree ± 1: $S^{i+j-1} \to S^{i+j-1}$. ∎

Example 5: Let $\beta\colon S^\bullet \to S^3$ such that $\Sigma^{2n}\beta \not\simeq *$ and $\Sigma^{2n+1}\beta \simeq *$ [Gra84, Theorem 12] or [Sta00, Corollary 9.2]. Denote by $w\colon S^4 \to S^3 \vee S^2$ the Whitehead bracket of the classes S^3 and S^2 and let $\gamma := w \circ \Sigma\beta$. Set $X = (S^3 \vee S^2) \cup_\gamma e^{\bullet+2}$. The method used in Example 4 gives $Q^{2n}cat(X) = 1$ and $Q^{2n-1}cat(X) = 2$.

The existence of $\beta\colon S^\bullet \to S^4$ such that $\Sigma^{2n-1}\beta \not\simeq *$ and $\Sigma^{2n}\beta \simeq *$ allows with the same process the construction of a space $X = (S^3 \vee S^3) \cup_\gamma e^{\bullet+2}$ such that $Q^{2n-1}cat(X) = 1$ and $Q^{2n-2}cat(X) = 2$.

We remark that the examples $X = Q_p$, $p > 2$, of N. Iwase [Iwa98] satisfy $2 = cat(X) = Q^1cat(X) > Q^2cat(X) = 1$. As for $X = Q_2$ of [Iwa98], it is such that $2 = cat(X) > Q^1cat(X) = 1$.

Example 6: For any $n \geq 1$ we denote by $X(n)$ a CW-complex which satisfies, as in Example 5, $Q^{2n-1}cat(X(n)) = 1$, $Q^{2n-2}cat(X(n)) = 2$ (by convention: $Q^0cat = cat$). Set $Y = \vee_{n\geq 1}X(n)$ and observe that Y (resp. $Y \times S^r$) dominates $X(n)$ (resp. $X(n) \times S^r$). We deduce from Corollary 2 and from [Iwa97] that Y is an **infinite** CW-complex such that $Qcat(Y) = 1$, $cat(Y) = 2$ and $cat(Y \times S^r) = cat(Y) + 1$ for any $r \geq 1$. This justifies the restriction to a **finite** complex in Problem 2.

Example 7: Denote by $\alpha_1(3) \in \pi_{2p}(S^3)$ a generator of the p-component and by $w\colon S^4 \to S^2 \vee S^3$ the Whitehead bracket. We deduce from Lemma 4 that $Q\mathcal{H}(w \circ \Sigma\alpha_1(3)) \not\simeq *$ and $\mathrm{Hur}\,\mathcal{H}(w \circ \Sigma\alpha_1(3)) \simeq *$. Therefore the space $X = (S^2 \vee S^3) \cup_{w\circ\Sigma\alpha_1(3)} e^{2p+2}$ satisfies $Qcat(X) = 2$ and $Mcat(X) = 1$.

We address now the relation between σcat and $Qcat$.

Example 8: (The Lemaire–Sigrist example revisited.) Denote by $w\colon S^5 \to \mathbb{C}P^2$ the attaching map of the top cell of $\mathbb{C}P^3$ and by $\gamma\colon S^6 \to \mathbb{C}P^2 \vee S^2$ the Whitehead bracket of w and S^2. Set $Z = (\mathbb{C}P^2 \vee S^2) \cup_\gamma e^7$. We claim that $Qcat(Z) = 3$ and $\sigma cat(Z) = \sigma^1 cat(Z) = e(Z) = 2$.

Observe that the rationalized space Z_0 satisfies $cat(Z_0) = Qcat(Z_0) = 3$ and $\sigma cat(Z_0) = e(Z_0) = 2$, [LS81]. We deduce that $3 \geq cat(Z) \geq Qcat(Z) \geq Qcat(Z_0) = 3$.

Consider the first Ganea space $G_1(X)$ associated to $X := \mathbb{C}P^2 \vee S^2$. From the decomposition $\Omega(\mathbb{C}P^2) \simeq S^1 \times \Omega(S^5)$, from B. Gray's formula [Gra71], and standard properties of Σ and Ω we see that $G_1(X)$ is a wedge of spheres. Among them we have $S^2_{(1)}$ corresponding to a generator of $\pi_2(\mathbb{C}P^2) = \mathbb{Z}$, S^5 corresponding to a generator of $\pi_5(\mathbb{C}P^2) = \mathbb{Z}$ and S^2. So we have a homotopy equivalence $G_1(X) \simeq S^2_{(1)} \vee S^5 \vee S^2 \vee \vee_i S^{n_i}$.

Let ι_1: $S^2_{(1)} \to G_1(X)$, ι: $S^2 \to G_1(X)$ and ι_5: $S^5 \to G_1(X)$ be the canonical inclusions. Let η: $S^3 \to S^2$ be the Hopf map. Then $q_1^X \circ \iota_1 \circ \eta$ is nullhomotopic and hence $\iota_1 \circ \eta$ is killed by the map $G_1(X) \to G_2(X)$. Hence we can find a section $X \to G_2(X)$. By $G_1(X) \to G_1(Z)$ the homotopy class of $[\iota_5, \iota]$ is mapped to an element $\tilde{\gamma}$ of $kernel(\pi_*(q_1^Z))$. Therefore $\tilde{\gamma}$ will be killed by $G_1(X) \to G_2(Z)$. Since $\Sigma\gamma$ and $\Sigma\tilde{\gamma}$ are both nullhomotopic, we can find a section $\Sigma Z \to \Sigma G_2(Z)$, i.e., $\sigma^1 cat(Z) \leq 2$.

Since $2 = \sigma^1 cat(Z_0) \leq \sigma^1 cat(Z)$ we get that $\sigma^1 cat(Z) = 2$.

Remark: We note that the notion of n-LS-fibration [ST97] does not allow an efficient use of Hopf invariants. For instance, the fact that id_{S^3}: $S^3 \to S^3$ is a 1-LS-fibration implies that a 1-LS fibration cannot bring a characterization of the category of $S^3 \cup_\alpha e^k$.

PROPOSITION 6: *For any space with two cells Problem 2 has a positive answer.*

Proof: Let $X = S^n \cup_\varphi e^p$. We may assume $cat(X) \geq 1$. If $cat(X) = 1$, then both statements are false. For $cat(X) = 2$ we refer to a result of [Iwa97]:

if $X = S^n \cup_\varphi e^p$ then $cat(X \times S^r) \leq cat(X)$ iff $\Sigma^r \mathcal{H}(\varphi) = 0$. ∎

Appendix A. Dror Farjoun's construction

In this paragraph we recall a construction from [DF96, Chapter 1.F.2]. Let λ: $\mathcal{S} \to \mathcal{S}$ be a *regular coaugmented functor* and π: $E \to B$ in $\mathcal{S}$ a fibration. We consider the **simplex category** Δ_B defined by:

– its objects are pairs $(\Delta[n], \sigma)$, $\sigma \in B_n$;

– a morphism α: $(\Delta[n], \sigma) \to (\Delta[m], \tau)$ is a simplicial map α: $\Delta[n] \to \Delta[m]$ such that $f_{\tau\circ\alpha} = f_\sigma$ where f_σ: $\Delta[n] \to B$ is the characteristic map of σ.

Denote by $\tilde{B}$: $\Delta_B \to \mathcal{S}$ the forgetful functor determined by $(\Delta[n], \sigma) \mapsto \Delta[n]$ and let $\tilde{E}$: $\Delta_B \to \mathcal{S}$ be the functor defined by the following pullback:

$$\begin{array}{ccc} \tilde{E}(\Delta[n], \sigma) & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \Delta[n] & \xrightarrow[f_\sigma]{} & B \end{array}$$

The projection $\tilde{E}(\Delta[n],\sigma) \to \Delta[n]$ defines a natural transformation $\tilde{E} \to \tilde{B}$. The homotopy colimits (in $\mathcal{S}$) of the functors $\tilde{B}$, $\lambda\circ\tilde{B}$, $\tilde{E}$ and $\lambda\circ\tilde{E}$ give a commutative diagram

$$\begin{array}{ccccc} hocolim\ \lambda\circ\tilde{E} & \longleftarrow & hocolim\ \tilde{E} & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ hocolim\ \lambda\circ\tilde{B} & \longleftarrow & hocolim\ \tilde{B} & \longrightarrow & B \end{array}$$

The functor $\overline{\lambda}$ is constructed with a homotopy pullback–pushout operation: P is the homotopy pullback (hpb) and $\overline{\lambda}(E)$ the homotopy pushout (hpo) defined in the following diagram:

$$\begin{array}{ccccc} & \swarrow & hocolim\ \tilde{E} & \longrightarrow & E \\ & & \downarrow & hpo & \downarrow \\ hocolim\ \lambda\circ\tilde{E} & \longleftarrow & P & \longrightarrow & \overline{\lambda}(E) \\ \downarrow & hpb & \downarrow & & \downarrow \\ hocolim\ \lambda\circ\tilde{B} & \longleftarrow & hocolim\ \tilde{B} & \longrightarrow & B \end{array}$$

This induces a factorization $E \to \overline{\lambda}(E) \to B$ of π. All diagrams

$$\begin{array}{ccc} \lambda(\tilde{E}(\Delta[m],\tau)) & \xrightarrow{\sim} & \lambda(\tilde{E}(\Delta[n],\sigma)) \\ \downarrow & & \downarrow \\ \lambda(\Delta[m],\tau) & \xrightarrow[\sim]{} & \lambda(\Delta[n],\sigma) \end{array}$$

are homotopy pullbacks. Hence by [Pup74] this implies:

Proposition 7 ([DF96, Chapter 1, Theorem F.3]): *For $b \in B$ let F be the fibre of π over b and $\overline{F}$ the homotopy fibre of $\overline{\lambda}(E) \to B$ over b. Then the induced map $F \to \overline{F}$ is naturally equivalent to the coaugmentation $F \to \lambda(F)$.*

Appendix B. Unpointed version of $\Omega^n\Sigma^n$

We now construct an unpointed version $Q^n\colon \mathcal{S} \to \mathcal{S}$ of $\Omega^n\Sigma^n\colon \mathcal{S}_* \to \mathcal{S}_*$ where $\mathcal{S}$ (resp. $\mathcal{S}_*$) is the convenient category of compactly generated (resp. well pointed compactly generated) spaces. For that we recall first the notion of unpointed suspension:

Definition 5: Let $I = [0,1]$. The **unreduced suspension** of $Y \in \mathcal{S}$ is $\widetilde{\Sigma}(Y) := (Y \times I)/\sim$, where $(y,0) \sim (y',0)$ and $(y,1) \sim (y',1)$ for any $y,\, y' \in Y$. By induction we define the n-**unreduced suspension** of $Y \in \mathcal{S}$ by $\widetilde{\Sigma^n}(Y) = \widetilde{\Sigma}\widetilde{\Sigma^{n-1}}(Y)$.

We will number the coordinates from right to left; i.e., an element of $\widetilde{\Sigma^n}(Y)$ is an equivalence class denoted by $[t_n, \ldots, t_1, y]$. Observe that we have a canonical map $j_n\colon \partial I^n \to \widetilde{\Sigma^n}(Y)$, $(t_n, \ldots, t_1) \mapsto [t_n, \ldots, t_1, y]$ (y arbitrary).

Definition 6: Given $Y \in \mathcal{S}$ we define $Q^n(Y)$ as the set of maps $\omega\colon I^n \to \widetilde{\Sigma^n}(Y)$ such that $\omega_{|\partial I^n} = j_n$. The map $c\colon Y \to Q^n(Y)$, $y \mapsto c(y)$, $c(y)(t_n, \ldots, t_1) = [t_n, \ldots, t_1, y]$ is a coaugmentation.

There are bonding maps $b_n\colon Q^n \to Q^{n+1}$ compatible with the coaugmentations given by $b_n(\omega)(t_{n+1}, \ldots, t_1) = [t_{n+1}, \omega(t_n, \ldots, t_1)]$ for $\omega \in Q^n(Y)$.

Set $Q(Y) := \varinjlim Q^n(Y)$.

Note that for $X \in \mathcal{S}_*$ the canonical map $\widetilde{\Sigma^n}(X) \to \Sigma^n(X)$ (where $\Sigma^n(X)$ is the reduced suspension) is a relative homeomorphism $(\widetilde{\Sigma^n}(X), \widetilde{\Sigma^n}(*)) \to (\Sigma^n(X), *)$ and that $\widetilde{\Sigma^n}(*)$ is contractible. Moreover, $\widetilde{\Sigma^n}(X) \to \Sigma^n(X)$ induces a map $Q^n(X) \to \Omega^n\Sigma^n(X)$.

PROPOSITION 8: (1) *The canonical map $Q^n(X) \to \Omega^n\Sigma^n(X)$ is a homotopy equivalence.*

(2) *For $Y,\, Z \in \mathcal{S}$ there is a canonical map $Q^n(Y) \times Z \to Q^n(Y \times Z)$ compatible with the coaugmentations.*

(3) *There is a natural transformation $m\colon Q^nQ^n \to Q^n$ such that Q^n together with c and m is a triple.*

Proof: (1) Note that for all $\omega \in Q^n(X)$ the restriction of ω to the boundary ∂I^n is equal to the restriction to ∂I^n of $I^n \to \widetilde{\Sigma^n}(*) \to \widetilde{\Sigma^n}(X)$. Thus dividing ∂I^{n+1} in two halves along an equator ∂I^n we obtain an element in $\Omega^n\widetilde{\Sigma^n}(X)$ by ω on one half and the composite $I^n \to \widetilde{\Sigma^n}(*) \to \widetilde{\Sigma^n}(X)$ on the other half. This gives an equivalence $Q^n(X) \to \Omega^n\widetilde{\Sigma^n}(X)$. Composing this map with $\Omega^n\widetilde{\Sigma^n}(X) \to \Omega^n\Sigma^n(X)$ we obtain the announced equivalence. Note that it is compatible with the bonding maps.

(2) We define $\eta\colon Q^n(Y) \times Z \to Q^n(Y \times Z)$ as follows. For $\omega \in Q^n(Y)$ write $\omega(t_n, \ldots, t_1) = [\tilde{t}_n, \ldots, \tilde{t}_1, \tilde{y}]$; then $\eta(\omega, z)(t_n, \ldots, t_1) = [\tilde{t}_n, \ldots, \tilde{t}_1, (\tilde{y}, z)]$. This definition does not depend on the choice of the representative in the class $\omega(t_n, \ldots, t_1)$ (because $\omega_{|\partial I^n}$ is the fixed canonical map j_n). One checks immediately that the map is compatible with the coaugmentations.

(3) We define $m\colon Q^nQ^n(Y) \to Q^n(Y)$ by the following device. Given $\omega\colon I^n \to \widetilde{\Sigma^n} Q^n(Y)$ write as above $\omega(t_n, \ldots, t_1) = [\tilde{t}_n, \ldots, \tilde{t}_1, \tilde{\omega}]$ with $\tilde{\omega} \in Q^n(Y)$. Then set $m(\omega)(t_n, \ldots, t_1) = \tilde{\omega}(\tilde{t}_n, \ldots, \tilde{t}_1)$. As above this definition does not depend on the choice of representative $[\tilde{t}_n, \ldots, \tilde{t}_1, \tilde{\omega}]$. A calculation shows that we have obtained a triple. ∎

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